

Optimal order quantity and pricing decisions in single-period inventory systems

L. Hakan Polatoglu*

Bilkent University, Department of Industrial Engineering, Ankara, Turkey

Abstract

In this paper, we consider simultaneous pricing and procurement decisions associated with a one-period pure inventory model under deterministic or probabilistic demand. We investigate the necessary and sufficient conditions for an (σ, Σ) type policy to be optimal for the determination of the procurement quantity. We also show how the corresponding optimal price can be obtained.

1. Introduction and literature review

In this paper, we study the optimal procurement and pricing decisions in a single-product, one-period pure inventory system. We view this model as a building block of the multi-period model and attempt to characterize an optimal one-period inventory control policy that would apply to the multi-period model under general assumptions.

Most inventory models are constructed under the assumption that the decisions of the vendor do not alter the demand pattern or the price structure in the market during the planning horizon. This assumption is approximated in a perfectly competitive market where there is no pricing decision to make for the individual vendor. Under imperfect competition, however, the vendor exercises a degree of monopoly power in the market and faces a downward sloping demand curve. He may set a price for his product but then he faces a demand level, governed by some probability distribution, the expected value of which is decreasing in price. At the beginning of the period, given the inventory position, his problem is to determine the procurement and pricing policies which jointly maximize the expected value of the one-period profit.

A number of special cases of this model have been studied in the literature. These differ essentially in the way the demand process is represented. In the *additive model*, $X(p) = \bar{X}(p) + \varepsilon$ where $X(p)$ is the demand when the price is p , $\bar{X}(p) = E[X(p)]$ and ε is a random variable with a known distribution and $E[\varepsilon] = 0$. In the *multiplicative model*, $X(p) = \bar{X}(p) \cdot \varepsilon$ where $E[\varepsilon] = 1$. In the *riskless model*, $X(p) = \bar{X}(p)$ so that the demand is represented by its expected value. This latter case serves both as a first order approximation and as a benchmark for the probabilistic version of the model. Note that while the demand variance is price-independent in the additive model, in the multiplicative model it is a decreasing function of price, under the (natural) assumption that $\bar{X}(p)$ is decreasing in price.

Whitin [1] appears to have been the first to link price theory and inventory control in a one-period model. Later, Mills [2] and Karlin and Carr [3] studied the additive model. They derived the necessary conditions for optimality and showed, under reasonable assumptions, that the optimal price under

*The author is grateful to Prof. I. Şahin (University of Wisconsin, Milwaukee) for his continual interest, helpful comments and timely encouragement during the preparation of this paper. The author is currently at the University of Wisconsin, Milwaukee, School of Business Administration, P.O. Box 742, Milwaukee, Wisconsin 53201, USA.

uncertainty is less than the optimal *riskless price*. This conclusion is reversed for the multiplicative model [3]. Zabel worked on the existence and uniqueness of the optimal solutions for the multiplicative [4] and additive [5] models. Young [6] also studied similar issues for a unified demand model in which $X(p)$ is given by a combination of the additive and multiplicative forms. These studies indicate that existence could be shown under restrictive assumptions on $X(p)$. Uniqueness, on the other hand, requires further restrictions, especially on the distribution of ε .

It has been common practice in demand modeling to express random demand as a combination of expected demand and a random term. The former has some form of price dependency while the latter is price independent. This synthesis has been used traditionally as a convenient tool to isolate the effects of uncertainty in the context of the *theory of the firm*. The disadvantage of this representation, however, is the structural restrictions it brings into the model. For instance, the additive model is restricted by a constant variance. Also it allows negative demands unless the price values are restricted. The multiplicative model implies the curious restriction that the demand equals the product of its expected value and a random term. As a result of this, variance of demand is the square of its expected value times the variance of the random term. Therefore, variance decreases at a rate faster than expected value and it approaches to zero at high prices.

We believe that there is a need to study the model under general demand uncertainty. It is essential to reveal the fundamental properties of the model, independent of the demand pattern. Especially, uniqueness conditions for optimality must be studied in a more general setting. In what follows, we introduce the basic model in Section 2 and develop and analyze it in Section 3. We then link the model to earlier studies in Section 4 by considering additive and multiplicative demand as special cases.

2. Basic model and assumptions

The vendor is to make the best procurement and pricing decisions to maximize his profit prior to the beginning of the period. Inventory level before ordering is i . The amount procured, if any, is $q-i$. A random demand X occurs during the period and at the end of the period the inventory level is reduced to $q-X$.

In this study, we consider the case where $i \geq 0$. For $i < 0$, the one-period problem is initiated with an unknown history. That is, the following questions can not be accounted for unless we make assumptions. (1) What fraction of the backlog do we have to satisfy? (2) At what price should we sell that fraction? (3) Do we deduct the backlog from the actual demand or not?

We assume that inventory costs are proportional to the period ending inventory level. We denote the unit holding, shortage and procurement costs by h , s and c , respectively. We also denote the fixed ordering cost by K .

In addition, we assume that the price is bounded from below and above by P_l and P_u , respectively, which are the price floor and price ceiling in a regulatory environment. We also assume that $P_u > c$ so that it is possible to make a profit by retailing.

In this study, we work with a finite demand process; that is:

$$0 \leq X_1(p) \leq X \leq X_2(p) < \infty$$

where $X_1(p)$ and $X_2(p)$ are the lower and upper bounds on X , respectively, which are differentiable functions of p . We are also given the demand distribution $F(x;p)$ which is defined over $x \in (-\infty, \infty)$ and $p \in [P_l, P_u]$. We shall restrict ourselves only to the continuous demand case, bearing in mind that similar analysis exists otherwise.

We assume that $\bar{X}(p)$ is a monotone decreasing function of p on $(0, \infty)$ (if p is confined to $[P_l, P_u]$, then we extend $\bar{X}(p)$ on $(0, P_l)$ and (P_u, ∞) by appropriate functions to satisfy the requirements without loss of generality). Moreover, we require that $\bar{X}(p)$ is $o(1/p)$ as $p \rightarrow 0^+$ and $p \rightarrow \infty$. This implies that the function $p \cdot \bar{X}(p)$ starts at zero, first increases and eventually dies away. This function, which is denoted by $R(p)$, is called the *riskless total revenue* by Mills [7]. $R(p)$ is a positive valued, finite and

differentiable function, which plays an important role in model development. It is shown in the Appendix that $R(p)$ is pseudoconcave on $(0, \infty)$ when $\bar{X}(p)$ is either a concave or convex decreasing function; it is also indicated that $R(p)$ is not pseudoconcave for all monotone decreasing $\bar{X}(p)$ functions. We assume that $R(p)$ is unimodal; hence, there exists a unique finite price which maximizes $R(p)$.

3. Mathematical model

In this section we develop and analyze the mathematical model under probabilistic demand for the determination of the optimal price and the beginning inventory level.

3.1 Optimization problem

Considering the representation introduced in Section 2, the profit function can be expressed as:

$$\Pi(p, q) = M(p, q) - \mathcal{K} \cdot \delta(q - i) \quad (1)$$

where $\delta(\cdot)$ is the Heavyside function and

$$M(p, q) = \begin{cases} p \cdot q - c \cdot (q - i) - s \cdot (X - q) & q \leq X \leq X_2(p) \\ p \cdot X - c \cdot (q - i) - h \cdot (q - X) & X_1(p) \leq X \leq q \end{cases} \quad (2)$$

where X is the random demand. We can write the expected profit as:

$$\bar{\Pi}(p, q) = E[\Pi(p, q)] = \bar{M}(p, q) - \mathcal{K} \cdot \delta(q - i) \quad (3)$$

where

$$\bar{M}(p, q) = E[M(p, q)] = p \cdot \bar{X}(p) - c \cdot (q - i) - L(p, q) \quad (4)$$

The first term in (4) is the *riskless total revenue* function. The second term is the procurement cost. The last term is the *expected loss* function which is given by

$$L(p, q) = (p + s) \cdot [\bar{X}(p) - q] + (p + s + h) \cdot \Theta(p, q) \quad (5)$$

where $\Theta(p, q)$ is the *expected leftovers*, i.e.

$$\Theta(p, q) = \int_{\bar{X}_1(p)}^q (q - x) \cdot f(x; p) dx = \int_{\bar{X}_1(p)}^q F(x; p) dx \quad (6)$$

We assume that $\Theta(p, q)$ is differentiable in p for $q \geq 0$. Moreover, we observe that $\Theta(p, q)$ satisfies

$$\Theta(p, q) \geq \max\{0, q - \bar{X}(p)\} \quad (7)$$

and it is a convex, nondecreasing and differentiable function of q for a given p .

From (4) and (5) it follows that

$$\bar{M}(p, q) = p \cdot [q - \Theta(p, q)] - c \cdot (q - i) - h \cdot \Theta(p, q) - s \cdot [\bar{X}(p) - (q - \Theta(p, q))] \quad (8)$$

Therefore, $\bar{M}(p, q)$ is the expected net revenue, less the procurement cost, less the expected holding cost, and less the expected shortage cost. At the expense of losing intuition about its terms, we shall refer to $\bar{M}(p, q)$ in the sequel in the following form.

$$\bar{M}(p, q) = (p + s - c) \cdot q - c \cdot i - \bar{X}(p) \cdot (p + s + h) \cdot \Theta(p, q) + c \cdot i \quad (9)$$

It is clear that, $\bar{M}(p, q)$ is continuous in p on $[P_1, P_u]$ and in q on $[0, \infty)$

Now, the optimization problem becomes

$$\bar{\Pi}(p^*, q^*) = \max_{p, q} \{\bar{\Pi}(p, q) : q \in [i, \infty), p \in [P_1, P_u]\} \quad (10)$$

where p^* and q^* are the optimal values of the decision variables p and q . For this problem we define the suboptimal function

$$\bar{M}^*(q) = \max\{\bar{M}(p, q) : p \in [P_l, P_u]\} = \bar{M}(p_q, q) \quad (11)$$

where p_q is the maximizer. Therefore, $\bar{M}^*(q)$ traces the *best price* trajectory over the q range. Moreover, since $\bar{M}(p, q)$ is continuous in p and q , $\bar{M}^*(q)$ becomes a continuous function of q .

In analyzing (10) and (11), we need to consider first and second degree partial derivatives of $\bar{M}(p, q)$ with respect to p and q , which are given by

$$\frac{\partial \bar{M}(p, q)}{\partial p} = q - s \cdot \frac{d\bar{X}(p)}{dp} - \Theta(p, q) - (p + s + h) \cdot \frac{\partial \Theta(p, q)}{\partial p} \quad (12)$$

$$\frac{\partial^2 \bar{M}(p, q)}{\partial p^2} = -s \cdot \frac{d^2 \bar{X}(p)}{dp^2} - 2 \cdot \frac{\partial \Theta(p, q)}{\partial p} - (p + s + h) \cdot \frac{\partial^2 \Theta(p, q)}{\partial p^2} \quad (13)$$

$$\frac{\partial^2 \bar{M}(p, q)}{\partial p \partial q} = 1 - F(q; p) - (p + s + h) \cdot \frac{\partial F(q; p)}{\partial p}$$

$$\frac{\partial \bar{M}(p, q)}{\partial q} = (p + s - c) - (p + s + h) \cdot F(q; p) \quad (14)$$

$$\frac{\partial^2 \bar{M}(p, q)}{\partial q^2} = -(p + s + h) \cdot f(q; p) \leq 0 \quad (15)$$

From (15) we conclude that $\bar{M}(p, q)$ is q -concave on $(0, \infty)$, which refers to the *newsboy problem* setting. On the other hand, (12) implies that p_q is independent of the procurement cost. In other words, the vendor is to maximize his expected profit given that he starts the period with q units. The price dependence of $\bar{M}(p, q)$, however, is not clear from (12) or (13).

If p_q is independent of q (a boundary point solution or a constant), then it follows from (15) that $\bar{M}^*(q)$ is concave at that q . However, if $p_q \in (P_l, P_u)$, then it must satisfy the first order condition

$$\partial \bar{M}(p, q) / \partial p|_{p_q} = 0$$

and the second order condition

$$\partial^2 \bar{M}(p, q) / \partial p^2|_{p_q} < 0$$

for a given q . Since $\bar{M}(p, q)$ has continuous partial derivatives, we can perform implicit differentiation on the first order condition to obtain

$$\frac{dp_q}{dq} = \frac{1 - F(q; p_q) - (p_q + s + h) \cdot \partial F(q; p) / \partial p|_{p_q}}{-\partial^2 \bar{M}(p, q) / \partial p^2|_{p_q}} \quad (16)$$

in which the denominator is always positive. Depending on the value of p_q and the price dependency of $F(\cdot; p)$ function, however, the numerator can be positive or negative. Thus, the sign of dp_q/dq is not clear.

Since dp_q/dq exists, we can write the first derivative of $\bar{M}^*(q)$ as

$$\frac{d\bar{M}^*(q)}{dq} = \frac{\partial \bar{M}(p, q)}{\partial q} + \frac{\partial \bar{M}(p, q)}{\partial p} \bigg|_{p_q} \cdot \frac{dp_q}{dq} \quad (17)$$

If $p_q \in (P_l, P_u)$, then $\partial \bar{M}(p, q) / \partial p|_{p_q} = 0$ otherwise $dp_q/dq = 0$. Therefore, in all combinations of right-hand and left-hand derivatives the second term in (17) vanishes. Consequently, we get

$$\frac{d\bar{M}^*(q)}{dq} = (p_q + s - c) - (p_q + s + h) \cdot F(q; p_q) \quad (18)$$

Furthermore, for $p_q \in (P_l, P_u)$, differentiating (18) with respect to q we obtain

$$\frac{d^2\bar{M}^*(q)}{dq^2} = \frac{dp_q}{dq} \cdot [1 - F(q; p_q)] - (p_q + s + h) \cdot \frac{dF(q; p_q)}{dq} \quad (19)$$

Noting that

$$\frac{dF(q; p_q)}{dq} = f(q; p_q) + \left. \frac{\partial F(q; p_q)}{\partial p} \right|_{p_q} \cdot \frac{dp_q}{dq} \quad (20)$$

we rewrite (19) as

$$\frac{d^2\bar{M}^*(q)}{dq^2} = - \left. \frac{\partial^2 \bar{M}(p, q)}{\partial p^2} \right|_{p_q} \cdot \left(\frac{dp_q}{dq} \right)^2 - (p_q + s + h) \cdot f(q; p_q) \quad (21)$$

The first term in (21) is always positive and the second is always negative. However, their relative magnitudes are not clear. Thus, convexity of $\bar{M}^*(q)$ is not evident from (21).

3.2. Existence problem

Intuitively, $\bar{M}^*(q)$ must have a peak on $[0, \infty)$. However, the existence of this point or, if it exists, its location is not immediately clear. In the following analysis, we shall identify two separate regions of q in which $\bar{M}^*(q)$ is monotone, then we shall prove the existence of its peak.

Lemma 1. $\forall q \in [0, X_1(P_u)]$, $\bar{M}^*(q)$ is a linear increasing function of q and $p_q = P_u$.

Proof. $\forall q \in [0, X_1(P_u)]$ we have $F(q; p_q) = 0$. Therefore, from (6), $\Theta(p_q, q) = 0$ and from (9) we obtain:

$$\begin{aligned} \bar{M}^*(q) &= \max\{(p + s - c) \cdot q - s \cdot \bar{X}(p) + c : p \in [P_l, P_u]\} \\ &= (P_u + s - c) \cdot q - s \cdot \bar{X}(P_u) + c \end{aligned} \quad (22)$$

which is a linear increasing function of q and $p_q = P_u$. \square

Lemma 1 indicates that, if we are sure that demand will exceed our stock, i.e. if $q \leq X_1(P_u)$, then we should charge the customers at the highest rate because we not only reduce shortages in this way but we also obtain the maximum unit profit.

If $X_1(P_u) = 0$, then the region indicated in Lemma 1 disappears and we lose the information about the slope of $\bar{M}^*(q)$ at $q = 0$. To account for this possibility, considering (18) and the fact that $0 \leq F(q; p_q) \leq 1$ we obtain:

$$-(h + c) \leq \frac{d\bar{M}^*(q)}{dq} \leq (p_o + s - c) \quad (23)$$

which gives the lower and upper limits of the rate of change of expected profit with respect to the beginning inventory level. It is now clear from (22) and (23) that at $q = 0$, $\bar{M}^*(q)$ increases at the maximum rate of $P_u + s - c$.

Lemma 2. $\forall q \in [X_2(P_l), \infty)$, $\bar{M}^*(q)$ is a linear decreasing function of q and p_q is a constant.

Proof. For $q > X_-(P_I)$ we have $F(q, p_q) = 1$. Therefore, from (6), $\Theta(p_q, q) = q - \bar{X}(p_q)$ and from (9) we obtain

$$\begin{aligned}\bar{M}^*(q) &= \max\{(p+h) \cdot \bar{X}(p), p \in [P_I, P_u]\} - (c+h) \cdot q + c \cdot t \\ &= (\bar{P}_h + h) \cdot \bar{X}(\bar{P}_h) - (c+h) \cdot q + c \cdot t\end{aligned}\quad (24)$$

where $\bar{P}_h = \min\{\max\{P_h, P_I\}, P_u\}$ and P_h is the maximizer of the pseudoconcave function $(p+h) \cdot \bar{X}(p)$. \square

We now establish the existence of \tilde{q} , where $\tilde{q} = \max\{\bar{M}^*(q) : q \in [0, \infty)\}$.

Theorem 1. $\exists \tilde{q} \in (X_-(P_u), X_2(P_I))$ such that $\bar{M}^*(q) \leq \bar{M}^*(\tilde{q}) \forall q \in [0, \infty)$

Proof. By Lemma 1, $\bar{M}^*(q)$ is a linear increasing function of q on $[0, X_1(P_u)]$ with a slope of $(P_u + s - c) > 0$. By Lemma 2, $\bar{M}^*(q)$ is a linear decreasing function of q on $[X_2(P_I), \infty)$ with a slope of $-(c+h) < 0$. From (23), $(P_u + s - c)$ and $-(c+h)$ are the largest and the smallest possible slopes of $\bar{M}^*(q)$, respectively. The proof follows. \square

Therefore, q must satisfy the first order optimality condition on $\bar{M}^*(q)$ which can be obtained from (18) as:

$$F(q, p_q) = \frac{p_q + s - c}{p_q + s + h} \quad (25)$$

RHS, the right hand side of (25), is a concave function of p_q . It becomes negative for $p_q < c - s$. It follows from (18) that, for those p_q values increasing, thus \tilde{q} can not be realized at any price level less than $c - s$. Alternatively, for $p_q \geq c - s$, RHS has values between 0 and 1, and we always have a solution for q given such RHS.

3.3 Unimodality

Unimodality of $\bar{M}^*(q)$ enables us to identify an (σ, Σ) type policy which may be employed in determining the optimal q . Moreover, in the multiperiod extension of the theory, this becomes an essential ingredient of the dynamic decision problem.

If the vendor administers his profit maximizing price as he starts with a stock size of q , then $F(q, p_q)$ represents the probability that there will be no shortage. Note that, $F(q, p_q)$ is a function of q only, where $F(q, p_q) = 0$ for $0 \leq q \leq X_1(P_u)$ and $F(q, p_q) = 1$ for $X_2(P_I) \leq q$. Therefore, $F(q, p_q)$ has to rise from 0 to 1 between minimum and maximum possible demand values. Meanwhile, it is clear from Lemma 1 and 2 that p_q should decrease from P_u to \bar{P}_h . If these changes occur monotonically, then there will be a unique first order q , which satisfies (25). That is, if $dF(q, p_q)/dq \geq 0$ and $dp_q/dq \leq 0$, then from (19) it follows that $\bar{M}^*(q)$ is concave. However, we can state a weaker condition by noting that it is sufficient to have $dp_q/dq \leq 0$ at $q = \tilde{q}$, provided that $dF(q, p_q)/dq \geq 0 \forall q$. That is,

$$\frac{dF(q, p_q)}{dq} \geq 0 \text{ and } \left. \frac{dp_q}{dq} \right|_q \leq 0 \Rightarrow \bar{M}^*(q) \text{ is unimodal} \quad (26)$$

Moreover from (16) and (25) we obtain

$$\left. \frac{dp_q}{dq} \right|_q \leq 0 \Leftrightarrow \left. \frac{\partial F(q, p_q)}{\partial p} \right|_{p_q} \geq \frac{h+c}{(\bar{p}+s+h)^2} \quad (27)$$

where $\bar{p} = p_q$ and we can employ (27) in (26). On the other hand, we realize that for unimodality of $\bar{M}^*(q)$ it is necessary and sufficient to have

$$\left. \frac{d^2 \bar{M}^*(q)}{dq^2} \right|_q \leq 0 \quad (28)$$

3.4 Optimal solution

If $\bar{M}^*(q)$ is unimodal, then from (10) it follows that q^* can be determined by an (σ, Σ) type policy operating on $\bar{M}^*(q)$, where $\Sigma = \bar{q}$ and $\Sigma = \min\{q, \bar{M}^*(q) = \bar{M}^*(\Sigma) - \mathcal{K}\}$. Consequently, the decision rule is $q^* = \Sigma$ if $\sigma < \sigma$, otherwise $q^* = 1$, and

$$p^* = \arg\max\{\bar{M}(p, q^*): p \in [P_l, P_u]\}$$

4. Special cases

In this section, first we consider the deterministic demand model (the *riskless* model introduced by Mills [2]) and establish its relation to the probabilistic model. Then, we analyze the additive and the multiplicative models. We provide the relationships that exist between the optimal prices of these models. Finally, under linear expected demand ($\bar{X}(p) = a - b \cdot p$, where $a, b > 0$ and $c < P_u < a/b$), we prove the unimodality of $\bar{M}^*(q)$ for uniformly distributed additive and for exponentially distributed multiplicative ε .

4.1 Deterministic model

In this part, we use the subscript "r" to denote the functions and variables of the riskless model. If there is no uncertainty in demand, then we have $X = \bar{X}(p)$. Under this specialization, leftovers are given by $\Theta_r(p, q) = \max\{0, q - \bar{X}(p)\}$, which is a continuous function. It is, however, non differentiable at the trajectory given by $q = \bar{X}(p)$.

In the following discussion, first we prove that $M_r^*(q)$ is unimodal, then we determine the optimal values of the decision variables, and finally we compare the deterministic and probabilistic profit functions.

Theorem 2. $M_r^*(q)$ is quasiconcave in q on $[0, \infty)$.

Proof For $q \leq \bar{X}(P_u)$ we have $\Theta_r(p, q) = 0$. Thus, from Lemma 1 it follows that $M_r^*(q)$ is a linear increasing function of q and $p_q = P_u$.

For $\bar{X}(P_u) < q$ we define \bar{p} such that $\bar{X}(\bar{p}) = \min\{q, \bar{X}(P_l)\}$. Therefore,

$$\Theta_r(p, q) = 0 \text{ if } P_l \leq p \leq \bar{p}$$

or

$$\Theta_r(p, q) = q - \bar{X}(p) \text{ if } \bar{p} \leq p \leq P_u$$

Under this setting, by Lemma 1 we have

$$\max\{M_r(p, q): P_l \leq p \leq \bar{p}\} = M_r(p, q)$$

Thus,

$$M_r^*(q) = \max\{M_r(p, q): \bar{p} \leq p \leq P_u\}$$

where $M_r(p, q) = (p+h) \cdot \bar{X}(p) - (c+h) \cdot q + c \cdot i$. We note that $(p+h) \cdot \bar{X}(p)$ is increasing on $[P_h, \bar{P}_h]$ and decreasing on $[\bar{P}_h, P_u]$. Moreover, $q \leq \bar{X}(\bar{P}_h) \Leftrightarrow \bar{p} \geq \bar{P}_h$.

It follows from the above discussion that

$$M_r^*(q) = \begin{cases} (P_u + s - c) \cdot q - s \cdot \bar{X}(P_u) + c \cdot i & q \leq \bar{X}(P_u) \\ (\bar{p} - c) \cdot q + c \cdot i & \bar{X}(P_c) \leq q \leq \bar{X}(\bar{P}_h) \\ -(c+h) \cdot q + (\bar{P}_h + h) \cdot \bar{X}(\bar{P}_h) + c \cdot i & \bar{X}(\bar{P}_h) \leq q \end{cases} \quad (29)$$

It is proven in the Appendix that $(\bar{p} - c) \cdot q$ is a pseudoconcave function of q on $(\bar{X}(P_u), \bar{X}(P_c))$. Thus, the result follows from (29).

From (29) it is clear that $\bar{p}_r = \bar{P}_r = \min\{\max\{P_c, P_u\}, P_u\}$, where P_c is the maximizer of the *riskless profit* function $(p-c) \cdot \bar{X}(p)$, and $\bar{q}_r = \bar{X}(\bar{P}_r)$.

We have $\Theta(p, q) \geq \Theta_r(p, q)$ from (7). Thus, it follows from (9) that $\bar{M}(p, q) \leq M_r(p, q)$ which implies $\bar{\Pi}(p, q) \leq \Pi_r(p, q)$. Also, comparing $\bar{M}^*(q)$ and $M^*(q)$ we conclude that $\bar{M}^*(q)$ remains below the quasiconcave function $M^*(q)$ and approaches it at both tails.

4.2 Additive model

Let $G(\cdot)$ be the distribution of ω , then $F(x; p) = G(x - \bar{X}(p))$ which implies that

$$\frac{\partial F(x; p)}{\partial p} = -f(x; p) \frac{d\bar{X}(p)}{dp} \text{ and } \frac{\partial \Theta(p, q)}{\partial p} = -F(q; p) \frac{d\bar{X}(p)}{dp}$$

With these results, the expressions (12) through (15) could be modified

Since \bar{p} is the optimal price at \bar{q} , it must satisfy the first order condition

$$\left. \frac{\partial \bar{M}(p, \bar{q})}{\partial p} \right|_{\bar{p}} = \bar{q} - \Theta(\bar{p}, \bar{q}) + (\bar{p} - c) \cdot \left. \frac{d\bar{X}(p)}{dp} \right|_{\bar{p}} = 0 \quad (30)$$

which implies that $\bar{p} > c$. By adding and subtracting $\bar{X}(\bar{p})$, (30) becomes

$$\bar{q} - \Theta(\bar{p}, \bar{q}) - \bar{X}(\bar{p}) + \left(\bar{X}(\bar{p}) + (\bar{p} - c) \cdot \left. \frac{d\bar{X}(p)}{dp} \right|_{\bar{p}} \right) = 0 \quad (31)$$

By definition, $\Theta(p, q) \geq q - \bar{X}(p)$. Therefore, the expression in the brackets, which is the derivative of the riskless profit function, evaluated at \bar{p} must be positive. Thus, we conclude that $c \leq \bar{p} \leq P_c$. This result was first proven by Mills [2] for a simple model. Karlin and Carr [3] showed that the same conclusion is true for the model we are studying by a different approach.

For a linear expected demand model and a uniform ε on $[-\lambda, \lambda]$ we have $F(q; p) = (q - a + b \cdot p + \lambda) / 2\lambda$ and $\Theta(p, q) = \lambda \cdot \Gamma(q; p)^2$ for all $q \in [\bar{X}(p) - \lambda, \bar{X}(p) + \lambda]$. Under this model, \bar{p} and \bar{q} must satisfy the first order conditions simultaneously (here, we ignore the presence of price bounds since the case of boundary solutions is trivial). These are given by

$$\frac{\bar{p} + s - c}{\bar{p} + s + h} = \frac{\bar{q} - a + b \cdot \bar{p} + \lambda}{2 \cdot \lambda} \quad (32)$$

and

$$\bar{q} - \frac{(\bar{q} - a + b \cdot \bar{p} + \lambda)^2}{4 \cdot \lambda} - b \cdot (\bar{p} - c) = 0 \quad (33)$$

Solving (32) and (33) for \bar{q} we get

$$\bar{q} = \lambda \cdot \left(\frac{\bar{p} + s - c}{\bar{p} + s + h} \right)^2 + b \cdot (\bar{p} - c) \quad (34)$$

Moreover, by substituting (34) in (32) and arranging the terms we obtain

$$2 \cdot (\bar{p} + s + h)^2 \cdot (P_c - \bar{p}) - \lambda \cdot \frac{(h+c)^2}{h} = 0 \quad (35)$$

which is a polynomial having a local maximum at $[2 \cdot P_c - (h+s)]/3$. It follows that this function has at least one and at most two positive roots. In addition, one of the roots is always located in the interval $([2 \cdot P_c - (h+s)]/3, P_c)$

Since the third critical point to make a local minimum does not exist, we conclude that $M^*(q)$ is unimodal.

4.3 Multiplicative model

By definition, $F(x;p) = G(x/\bar{X}(p))$, which implies that

$$\frac{\partial F(x;p)}{\partial p} = -x \cdot f(x;p) \cdot (1/\bar{X}(p)) \cdot \frac{d\bar{X}(p)}{dp}$$

and

$$\frac{\partial \Theta(p,q)}{\partial p} = -[q \cdot F(q;p) - \Theta(p,q)] \cdot (1/\bar{X}(p)) \cdot \frac{d\bar{X}(p)}{dp}$$

With these results, the expressions (12) through (15) could be modified.

Evaluating (12) at p_q and arranging terms we get

$$\begin{aligned} \frac{\partial \bar{M}(p,q)}{\partial p} \Big|_{p_q} &= q \cdot [1 - F(q;p_q)] + \frac{q \cdot F(q;p_q) - \Theta(p_q,q)}{\bar{X}(p_q)} \cdot \left\{ \bar{X}(p) + (p+h) \cdot \frac{d\bar{X}(p)}{dp} \right\} \Big|_{p_q} \\ &\quad - s \cdot \frac{d\bar{X}(p)}{dp} \Big|_{p_q} \cdot \frac{\bar{X}(p_q) - q \cdot F(q;p_q) + \Theta(p_q,q)}{\bar{X}(p_q)} = 0 \end{aligned} \quad (36)$$

Since $\Theta(p,q) \geq q \cdot \bar{X}(p)$, we have $\Theta(p,q) + \bar{X}(p) - q \cdot F(q;p) \geq 0$. Thus, the first and the third terms in (36) are positive. Moreover, we note that $q \cdot F(q;p) - \Theta(p,q) \geq 0$. Therefore, (36) implies that

$$\left\{ \bar{X}(p) + (p+h) \cdot \frac{d\bar{X}(p)}{dp} \right\} \Big|_{p_q} \leq 0 \Leftrightarrow p_q \geq \bar{P}_h$$

Furthermore, evaluating (12) at \bar{p} and \bar{q} and rearranging the terms we obtain

$$\begin{aligned} \frac{\bar{q}}{\bar{X}(\bar{p})} \cdot \left\{ \bar{X}(p) + (p-c) \cdot \frac{d\bar{X}(p)}{dp} \right\} \Big|_{\bar{p}} - \frac{\Theta(\bar{p},\bar{q})}{\bar{X}(\bar{p})} \cdot \left\{ \bar{X}(p) + (p+h) \cdot \frac{d\bar{X}(p)}{dp} \right\} \Big|_{\bar{p}} \\ - s \cdot \frac{d\bar{X}(p)}{dp} \Big|_{\bar{p}} \cdot \frac{\bar{X}(\bar{p}) - \bar{q} + \Theta(\bar{p},\bar{q})}{\bar{X}(\bar{p})} = 0 \end{aligned} \quad (37)$$

The second term is positive, since $\bar{p} \geq \bar{P}_h$ and so is the third term. Therefore, we must have

$$\left\{ \bar{X}(p) + (p-c) \cdot \frac{d\bar{X}(p)}{dp} \right\} \Big|_{\bar{p}} \leq 0 \Leftrightarrow \bar{p} \geq \bar{P}_c > c$$

This result is the same as Karlin and Carr's [3] conclusion, which was proved by a different approach than ours.

If $\bar{X}(p)$ is linear and ε has an exponential distribution, then the unimodality condition (28) reduces to

$$2 \cdot \bar{p}^2 + (3 \cdot h + 4 \cdot s - c) \cdot \bar{p} + 2 \cdot (s + h) \cdot (s - c) - (h + c) \cdot \frac{a}{b} \geq 0 \quad (38)$$

The minimizer of the quadratic function in (38) is $-(3 \cdot h + 4 \cdot s - c)/4$ which is less than c ; hence, it is also less than P_c . It can be shown that if $a/b > c$, then the value of the quadratic function evaluated at P_c is positive. Since $\bar{p} > P_c$, this result implies condition (38). Therefore, $\bar{M}^*(q)$ is unimodal for the exponential multiplicative demand model. Zabel [4] arrived at the same conclusion, under some restrictions for the case where $s=0$.

5. Conclusions

There are analytical difficulties in verifying the unimodality of $\bar{M}^*(q)$ in a given problem. These arise mainly because p_q or \bar{q} can not be explicitly evaluated. One possibility is to make simplifying assumptions so that analytical difficulties can be overcome. However, there is no major practical difficulty in testing these conditions numerically. We refer the reader to [8] for numerical examples.

Since pricing decision affects the period ending inventory level, optimality of (σ, S) type policies for the multi-period model does not follow from the analysis of the one-period model. These issues are under current investigation.

6. Appendix

For $R(p) = p \cdot \bar{X}(p)$ we have

$$R'(p) = \bar{X}(p) + p \cdot \bar{X}'(p) \quad (39)$$

$$R''(p) = 2 \cdot \bar{X}'(p) + p \cdot \bar{X}''(p) \quad (40)$$

Lemma A1. $R(p)$ is not pseudoconcave for all monotone decreasing $\bar{X}(p)$ functions.

Proof. If we let $\bar{X}(p) = 600 \cdot e^{-0.15 \cdot p} + 1.5 \cdot \sin(2 \cdot \pi \cdot p)$, which is a monotone decreasing function of p on $(0, 8)$, then $R(p)$ is not a pseudoconcave function on $(0, 8)$. \square

Lemma A2. If $\bar{X}(p)$ is a convex decreasing function, then $R(p)$ is pseudoconcave on $(0, \infty)$.

Proof. Since $\bar{X}(p)$ is a convex decreasing function, $\forall p, p_1 \in (0, \infty)$ we have

$$\bar{X}(p_1) - \bar{X}(p) \geq (p_1 - p) \cdot \bar{X}'(p_1) \quad (41)$$

By definition, $R(p)$ will be pseudoconcave at $p_1 \in (0, \infty)$ if it is differentiable at p_1 and

$$R'(p_1) \cdot (p - p_1) \leq 0 \Rightarrow R(p) \leq R(p_1), \quad \forall p \in (0, \infty) \quad (42)$$

Using (39) and (41) in (42) and arranging terms we get

$$R'(p_1) \cdot (p - p_1) \leq 0 \Rightarrow R(p) + (p - p_1) \cdot [\bar{X}(p_1) - \bar{X}(p)] \leq R(p_1) \rightarrow R(p) \leq R(p_1)$$

Since p_1 was arbitrary the proof is valid for all $p_1 \in (0, \infty)$. \square

Theorem A1. If $\bar{X}(p)$ is a convex or concave decreasing function, then $R(p)$ is pseudoconcave on $(0, \infty)$.

Proof. If $\bar{X}(p)$ is concave, then from (4C), it follows that $R(p)$ is concave on $(0, \infty)$. Also by Lemma A2, $R(p)$ is pseudoconcave on $(0, \infty)$ for a convex decreasing function. \square

Corollary A1. The function $T(p) = (p+a) \cdot \bar{X}(p)$ is pseudoconcave on $(0, \infty)$, where $a \in \mathcal{R}$.

Proof. Making a coordinate change by $p_2 \leftarrow p+a$ and introducing the function $Y(p_2) = \bar{X}(p_2-a)$ we obtain $T(p) = (p+a) \cdot \bar{X}(p) = p_2 \cdot Y(p_2)$. By Theorem A1, $p_2 \cdot Y(p_2)$ is pseudoconcave on (a, ∞) which implies that $T(p)$, being a translation of $p_2 \cdot Y(p_2)$, is pseudoconcave on $(0, \infty)$. \square

Corollary A2. $(p-c) \cdot q$ is a pseudoconcave function of q on $(\bar{X}(P_u), \bar{X}(P_l))$, where $q = \bar{X}(p)$.

Proof. $\bar{X}(p)$ is a decreasing function of p . Therefore, its inverse, $\bar{X}^{-1}(q)$, is decreasing on $(\bar{X}(P_u), \bar{X}(P_l))$. By Theorem A1, $q \cdot \bar{X}^{-1}(q)$ is pseudoconcave on $(\bar{X}(P_u), \bar{X}(P_l))$. Thus, $q \cdot \bar{X}^{-1}(q) - c \cdot q = (p-c) \cdot q$ is also pseudoconcave on $(\bar{X}(P_u), \bar{X}(P_l))$. \square

References

1. Whittin, T M., 1955. Inventory control and price theory. *Management Sci.*, 2. 61-68.
2. Mills, E.S., 1959. Uncertainty and price theory. *Quart. J. Econom.*, 73: 116-130.
3. Karlin, S and Carr, R.C., 1962. Prices and optimal inventory policy. In: K.J. Arrow, S. Karlin and H. Scarf (Eds.), *Studies in Applied Probability and Management Science*, Chapter 10. Stanford University Press, Stanford, CA.
4. Zabel, E., 1970. Monopoly and uncertainty. *Rev. Economic Stud.*, 37: 205-219.
5. Zabel, E., 1972. Multiperiod monopoly under uncertainty. *J. Econom. Stud.*, 5: 524-536.
6. Young, L., 1978. Price, inventory and the structure of uncertain demand. *New Zealand J. Oper. Res.*, 6: 157-177.
7. Mills, E S., 1962. *Price, Output, and Inventory Policy*. Wiley, New York.
8. Lau, A Hing-Ling and Lau, Hon-Shiang, 1988. The newsboy problem with price-dependent demand distribution. *IIE Trans.*, 20. 168-175.